

PREASYMPTOTICS AND ASYMPTOTICS OF APPROXIMATION NUMBERS OF ANISOTROPIC SOBOLEV EMBEDDINGS

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ABSTRACT. In this paper, we obtain the preasymptotic and asymptotic behavior and strong equivalences of the approximation numbers of the embeddings from the anisotropic Sobolev spaces $W_2^{\mathbf{R}}(\mathbb{T}^d)$ to $L_2(\mathbb{T}^d)$. We also get the preasymptotic behavior of the approximation numbers of the embeddings from the limit spaces $W_2^\infty(\mathbb{T}^d)$ of the anisotropic Sobolev spaces $W_2^{\mathbf{R}}(\mathbb{T}^d)$ to $L_2(\mathbb{T}^d)$. We show that both the above embedding problems are intractable and do not suffer from the curse of dimensionality.

1. INTRODUCTION

Let X, Y be two Banach spaces. For a bounded linear operator $T : X \rightarrow Y$, the approximation numbers of it are defined as

$$\begin{aligned} a_n(T : X \rightarrow Y) &:= \inf_{\text{rank } A < n} \sup_{\|x\|_X \leq 1} \|Tx - Ax\|_Y \\ &= \inf_{\text{rank } A < n} \|T - A : X \rightarrow Y\|, \quad n \in \mathbb{N}_+. \end{aligned}$$

They describe the best approximation of T by finite rank operators.

Recently, the papers [3, 4] considered sharp constants of the approximation numbers and tractability of the embeddings from the Sobolev spaces of isotropic and dominating mixed (fractional) smoothness for various equivalent norms including the classical one on the d -dimensional torus \mathbb{T}^d to the L_2 space. Specially, the authors of [3, 2] and [4] obtained the asymptotic and preasymptotic behavior of the approximation numbers of the isotropic Sobolev embeddings and the mixed order Sobolev embeddings. We note that there are another Sobolev spaces—anisotropic Sobolev spaces, which may be viewed as generalization of the isotropic Sobolev spaces. In this paper, we investigate the preasymptotic and asymptotic behavior of the approximation numbers of the anisotropic Sobolev embeddings

$$(1.1) \quad I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \longrightarrow L_2(\mathbb{T}^d), \quad \mathbf{R} = (R_1, R_2, \dots, R_d) \in \mathbb{R}_+^d,$$

and the limit space embeddings of the anisotropic Sobolev spaces

$$(1.2) \quad I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d),$$

where I_d is the identity (embedding) operator, and $W_2^{\mathbf{R}}(\mathbb{T}^d)$ and $W_2^\infty(\mathbb{T}^d)$ are the anisotropic Sobolev spaces and their limit spaces whose definitions will be given in Sections 2.2 and 2.3.

Key words and phrases. Sharp constants; Asymptotics; Preasymptotics; Approximation numbers; Tractability, Anisotropic Sobolev spaces.

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In recent years, there has been an increasing interest in multivariate computational problems which are defined on classes of functions depending on d variables, since many problems, e.g., in finance or quantum chemistry, are modeled in associated function spaces on high-dimensional domains. So far, many authors have contributed to the subject, see for instance the monographs by Temlyakov [10] and the references therein. In [10, Chapter 2, Theorems 4.1, 4.2], the following two-sided estimate can be found:

$$(1.3) \quad c(\mathbf{R}, d)n^{-g(\mathbf{R})} \leq a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \leq C(\mathbf{R}, d)n^{-g(\mathbf{R})}, \quad n \in \mathbb{N}_+,$$

where $g(\mathbf{R}) = \frac{1}{1/R_1 + \dots + 1/R_d}$, and the constants $c(\mathbf{R}, d)$ and $C(\mathbf{R}, d)$, only depending on d and \mathbf{R} , were not explicitly determined. When $R_1 = R_2 = \dots = R_d = s > 0$, $W_2^{\mathbf{R}}(\mathbb{T}^d)$ recedes to the natural isotropic Sobolev space $H^{s, 2s}(\mathbb{T}^d)$, $g(\mathbf{R}) = s/d$, where the definition of the Sobolev space $H^{s, r}(\mathbb{T}^d)$ ($s > 0$, $0 < r \leq \infty$) will be given in Section 2.1. When $\min\{R_1, \dots, R_d\} \rightarrow +\infty$, $W_2^{\mathbf{R}}(\mathbb{T}^d)$ tends to $W_2^\infty(\mathbb{T}^d)$ in the sense of the set limit.

Our main focus is to clarify, for arbitrary but fixed \mathbf{R} , the dependence of these constants on d . Surprisingly, for sufficiently large n , say $n > 3^d$, it turns out that the optimal constants decay polynomially in d , i.e.,

$$c(\mathbf{R}, d) \asymp C(\mathbf{R}, d) \asymp d^{-1/2},$$

where equivalent constants depend only on $\max\{R_1, \dots, R_d\}$ and $\min\{R_1, \dots, R_d\}$. Specially, we obtain the strong equivalence of the approximation numbers $a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d))$ as $n \rightarrow \infty$.

For small n , $1 \leq n \leq 3^d$, we also determine explicitly how the approximation numbers $a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d))$ and $a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d))$ behave preasymptotically. We emphasize that the preasymptotic behavior of $a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d))$ is completely different from its asymptotic behavior. However, the preasymptotic behavior of $a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d))$ coincides with the one of $a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d))$. Our main results will be in full analogy with those of [3, 2] for the case of $H^{s, 2s}(\mathbb{T}^d)$.

Finally we consider weak tractability results for the approximation problem of the anisotropic Sobolev embeddings (1.1) and (1.2). Based on results of [3, 2] and preasymptotic behavior of $a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d))$, we show that the approximation problems (1.1) and (1.2) are intractable and do not suffer from the curse of dimensionality.

The paper is organized as follows. In Section 2 we give definitions of the isotropic Sobolev spaces with various equivalent norms, the anisotropic Sobolev spaces, the limit spaces $W_2^\infty(\mathbb{T}^d)$, and tractability, and then state out main results. In the final Section 3 we prove the main results.

2. PRELIMINARIES AND MAIN RESULTS

2.1. Isotropic Sobolev spaces.

For $d \in \mathbb{N}_+$, $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, we set $|\mathbf{x}|_p = \left(\sum_{j=1}^d |x_j|^p\right)^{1/p}$ for $0 < p < \infty$, and $|\mathbf{x}|_\infty = \max_{1 \leq j \leq d} |x_j|$ for $p = \infty$. In what follows \mathbb{T} denotes the torus, i.e., $\mathbb{T} = [0, 2\pi]$, where the endpoints of the interval are identified, and \mathbb{T}^d stands for the d -dimensional torus. We equip \mathbb{T}^d with the normalized Lebesgue measure $(2\pi)^{-d}d\mathbf{x}$. Consequently, $\{e^{i\mathbf{k}\mathbf{x}} : \mathbf{k} \in \mathbb{Z}^d\}$ is an orthonormal basis in $L_2(\mathbb{T}^d)$, where

$\mathbf{k}\mathbf{x} = \sum_{j=1}^d k_j x_j$. The Fourier coefficients of a function $f \in L_1(\mathbb{T}^d)$ are defined as

$$\hat{f}(\mathbf{k}) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-i\mathbf{k}\mathbf{x}} d\mathbf{x}, \quad \mathbf{k} \in \mathbb{Z}^d.$$

For $0 < s < \infty$ and $0 < r \leq \infty$ we denote by $H^{s,r}(\mathbb{T}^d)$ the isotropic Sobolev space formed by all $f \in L_2(\mathbb{T}^d)$ having a finite norm

$$(2.1) \quad \|f|H^{s,r}(\mathbb{T}^d)\| = \left(\sum_{\mathbf{k} \in \mathbb{Z}^d} (1 + \sum_{j=1}^d |k_j|^r)^{2s/r} |\hat{f}(\mathbf{k})|^2 \right)^{1/2}.$$

Clearly, for the fixed isotropic smoothness index $s > 0$, all these norms are equivalent, whence all spaces $H^{s,r}(\mathbb{T}^d)$ with $0 < r \leq \infty$ coincide. The superscript r just indicates which norm we are considering. For integer smoothness $s = m \in \mathbb{N}_+$, the most natural norms are those with $r = 2$ and $r = 2m$. Indeed, let $D^\alpha f$ be the distributional derivative of f of order $\alpha = (\alpha_1, \dots, \alpha_d)$. The natural isotropic Sobolev space $W_2^m(\mathbb{T}^d)$ is defined as

$$W_2^m(\mathbb{T}^d) := \left\{ f \mid \|f|W_2^m(\mathbb{T}^d)\| = \left(\|f|L_2(\mathbb{T}^d)\|^2 + \sum_{j=1}^d \left\| \frac{\partial^m f}{\partial x_j^m} \right\|_{L_2(\mathbb{T}^d)}^2 \right)^{1/2} < \infty \right\}.$$

If $r = 2m$, one has equality

$$\|f|H^{m,2m}(\mathbb{T}^d)\| = \|f|W_2^m(\mathbb{T}^d)\|.$$

The classical isotropic Sobolev space $H^m(\mathbb{T}^d)$ is defined as

$$H^m(\mathbb{T}^d) := \left\{ f \mid \|f|H^m(\mathbb{T}^d)\| = \left(\sum_{\alpha \in \mathbb{N}^d, |\alpha|_1 \leq m} \|D^\alpha f|L_2(\mathbb{T}^d)\|^2 \right)^{1/2} < \infty \right\}.$$

As shown in [3], one has

$$(2.2) \quad \frac{1}{\sqrt{m!}} \|f|H^{m,2}(\mathbb{T}^d)\| \leq \|f|H^m(\mathbb{T}^d)\| \leq \|f|H^{m,2}(\mathbb{T}^d)\|.$$

Note that the equivalence constants depend only on the smoothness index m , but not on the dimension d .

2.2. Anisotropic Sobolev spaces.

For $f \in L_2(\mathbb{T}^d)$ and $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$, let $\frac{\partial^{R_j}}{\partial x_j^{R_j}} f$ be the R_j -order partial derivative of f with respect to x_j in the sense of Weyl defined by

$$\frac{\partial^{R_j}}{\partial x_j^{R_j}} f(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^d} (ik_j)^{R_j} \hat{f}(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}}, \quad (ik_j)^{R_j} = |k_j|^{R_j} \exp\left(\frac{R_j \pi i}{2} \text{sign } k_j\right).$$

The anisotropic Sobolev space $W_2^{\mathbf{R}}(\mathbb{T}^d)$ is defined by

$$W_2^{\mathbf{R}}(\mathbb{T}^d) := \left\{ f \in L_2(\mathbb{T}^d) \mid \frac{\partial^{R_j}}{\partial x_j^{R_j}} f \in L_2(\mathbb{T}^d), \quad j = 1, \dots, d \right\}$$

with norm

$$\begin{aligned}\|f|W_2^{\mathbf{R}}(\mathbb{T}^d)\| &= \left(\|f|L_2(\mathbb{T}^d)\| + \sum_{j=1}^d \left\| \frac{\partial^{R_j}}{\partial x_j^{R_j}} f |L_2(\mathbb{T}^d)\right\|^2 \right)^{1/2} \\ &= \left(\sum_{\mathbf{k} \in \mathbb{Z}^d} \left(1 + \sum_{j=1}^d |k_j|^{2R_j} \right) |\hat{f}(\mathbf{k})|^2 \right)^{1/2}.\end{aligned}$$

It is well known that $W_2^{\mathbf{R}}(\mathbb{T}^d)$ is a Hilbert space. If $R_1 = \dots = R_d = s > 0$, the anisotropic Sobolev space $W_2^{\mathbf{R}}(\mathbb{T}^d)$ recedes to the isotropic Sobolev space $H^{s,2s}(\mathbb{T}^d)$. Hence, the anisotropic Sobolev spaces are generalization of the isotropic Sobolev spaces. For integer smoothness $\mathbf{R} = \mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}_+^d$, $W_2^{\mathbf{m}}(\mathbb{T}^d)$ is just the natural anisotropic Sobolev space. The classical anisotropic Sobolev space $H^{\mathbf{m}}(\mathbb{T}^d)$ is defined by

$$H^{\mathbf{m}}(\mathbb{T}^d) := \left\{ f \mid \|f|H^{\mathbf{m}}(\mathbb{T}^d)\| = \left(\sum_{\alpha \in \mathbb{N}^d, \frac{\alpha_1}{m_1} + \dots + \frac{\alpha_d}{m_d} \leq 1} \|D^\alpha f|L_2(\mathbb{T}^d)\|^2 \right)^{1/2} < \infty \right\}.$$

When $m_1 = \dots = m_d = m$, the natural and classical anisotropic Sobolev spaces $W_2^{\mathbf{m}}(\mathbb{T}^d)$ and $H^{\mathbf{m}}(\mathbb{T}^d)$ recede to the natural and classical (isotropic) Sobolev spaces $W_2^m(\mathbb{T}^d)$ and $H^m(\mathbb{T}^d)$. It is easily seen that

$$\|f|W_2^{\mathbf{m}}(\mathbb{T}^d)\| \leq \|f|H^{\mathbf{m}}(\mathbb{T}^d)\| \leq C_{d,\mathbf{m}} \|f|W_2^{\mathbf{m}}(\mathbb{T}^d)\|,$$

where $C_{d,\mathbf{m}}$ is a positive constant depending only on d and \mathbf{m} . Due to (2.2), one may conjecture that for general $\mathbf{m} \in \mathbb{N}_+^d$,

$$c_{\mathbf{m}} \|f|H^{\mathbf{m}}(\mathbb{T}^d)\|^2 \leq \sum_{\mathbf{k} \in \mathbb{Z}^d} \left(1 + \sum_{j=1}^d |k_j|^{2m_j/p_0} \right)^{p_0} |\hat{f}(\mathbf{k})|^2 \leq C_{\mathbf{m}} \|f|H^{\mathbf{m}}(\mathbb{T}^d)\|^2$$

holds for some p_0 , $c_{\mathbf{m}}$, $C_{\mathbf{m}}$, where the constants $c_{\mathbf{m}}$ and $C_{\mathbf{m}}$ depend only on \mathbf{m} . However, this is not true. Indeed, for any positive numbers $p_0, c_{\mathbf{m}}$, $C_{\mathbf{m}}$, the above inequality is not valid. In this paper, we consider only the space $W_2^{\mathbf{R}}(\mathbb{T}^d)$.

2.3. Limit space of anisotropic (or isotropic) Sobolev spaces.

Let X_j be Banach spaces. We define $\bigwedge_{j=1}^{\infty} X_j$ to be the space of all elements of $\bigcap_{j=1}^{\infty} X_j$ for which $\sup_{1 \leq j < \infty} \|x|X_j\| < \infty$, i.e.,

$$\bigwedge_{j=1}^{\infty} X_j = \left\{ x \in \bigcap_{j=1}^{\infty} X_j \mid \|x\| \bigwedge_{j=1}^{\infty} X_j = \sup_{1 \leq j < \infty} \|x|X_j\| < \infty \right\}.$$

In this paper, we consider the space $W_2^{\infty}(\mathbb{T}^d) = \bigwedge_{\mathbf{m} \in \mathbb{N}_+^d} W_2^{\mathbf{m}}(\mathbb{T}^d)$. Clearly, $W_2^{\infty}(\mathbb{T}^d)$ may be viewed as the limit space of the anisotropic Sobolev spaces $W_2^{\mathbf{R}}(\mathbb{T}^d)$. Indeed, when $\min\{R_1, \dots, R_d\} \rightarrow +\infty$, $W_2^{\mathbf{R}}(\mathbb{T}^d)$ tends to $W_2^{\infty}(\mathbb{T}^d)$ in the sense of the set limit.

Note that if $f \in W_2^\infty(\mathbb{T}^d)$ and $\mathbf{k} \in \mathbb{Z}^d$, $|k_i| \geq 2$ for some $i \in \{1, 2, \dots, d\}$, then $\hat{f}(\mathbf{k}) = 0$. It follows that

$$W_2^\infty(\mathbb{T}^d) = \left\{ f \mid f(x) = \sum_{\mathbf{k} \in \{-1, 0, 1\}^d} \hat{f}(\mathbf{k}) e^{-i\mathbf{k}\mathbf{x}}, \right.$$

$$\left. \|f\|_{W_2^\infty(\mathbb{T}^d)} = \left(\sum_{\mathbf{k} \in \{-1, 0, 1\}^d} \left(1 + \sum_{j=1}^d |k_j|\right) |\hat{f}(\mathbf{k})|^2 \right)^{1/2} \right\}.$$

Clearly, $W_2^\infty(\mathbb{T}^d)$ is a Hilbert space with $\dim(W_2^\infty(\mathbb{T}^d)) = 3^d$. We also note that

$$W_2^\infty(\mathbb{T}^d) = \bigcap_{m=1}^{\infty} W_2^m(\mathbb{T}^d).$$

Remark 2.1. We note that the space $\bigwedge_{m=1}^{\infty} H^m(\mathbb{T}^d)$ is just the space of constants and its dimension is 1. This means the investigation of $\bigwedge_{m=1}^{\infty} H^m(\mathbb{T}^d)$ is meaningless.

2.4. General notions of tractability.

Recently, there has been an increasing interest in d -variate computational problems with large or even huge d . Such problems are usually solved by algorithms that use finitely many information operations. The information complexity $n(\varepsilon, d)$ is defined as the minimal number of information operations which are needed to find an approximating solution to within an error threshold ε . A central issue is the study of how the information complexity depends on ε^{-1} and d . Such problem is called the tractable problem. Nowadays tractability of multivariate problems is a very active research area (see [5, 6, 7] and the references therein).

Let H_d and G_d be two sequences of Hilbert spaces and for each $d \in \mathbb{N}_+$, F_d be the unit ball of H_d . Assume a sequence of bounded linear operators (solution operators)

$$S_d : H_d \rightarrow G_d$$

for all $d \in \mathbb{N}_+$. For $n \in \mathbb{N}_+$ and $f \in F_d$, $S_d f$ can be approximated by algorithms

$$A_{n,d}(f) = \Phi_{n,d}(L_1(f), \dots, L_n(f)),$$

where L_j , $j = 1, \dots, n$ are continuous linear functionals on F_d which are called general information, and $\Phi_{n,d} : \mathbb{R}^n \rightarrow G_d$ is an arbitrary mapping. The worst case error $e(A_{n,d})$ of the algorithm $A_{n,d}$ is defined as

$$e(A_{n,d}) = \sup_{f \in F_d} \|S_d(f) - A_{n,d}(f)\|_{G_d}.$$

Furthermore, we define the n th minimal worst-case error as

$$e(n, d) = \inf_{A_{n,d}} e(A_{n,d}),$$

where the infimum is taken over all algorithms using n information operators L_1, L_2, \dots, L_n . For $n = 0$, we use $A_{0,d} = 0$. The error of $A_{0,d}$ is called the initial error and is given by

$$e(0, d) = \sup_{f \in F_d} \|S_d f\|_{G_d}.$$

The n th minimal worst-case error $e(n, d)$ with respect to arbitrary algorithms and general information in the Hilbert setting is just the $n + 1$ -approximation number $a_{n+1}(S_d : H_d \rightarrow G_d)$ (see [5, p. 118]), i.e.,

$$e(n, d) = a_{n+1}(S_d : H_d \rightarrow G_d).$$

In this paper, we consider the embedding operators $S_d = I_d$ (formal identity operators) with $e(0, d) = \|I_d\| = 1$. In other words, the normalized error criterion and the absolute error criterion coincide. For $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}_+$, let $n(\varepsilon, d)$ be the information complexity which is defined as the minimal number of continuous linear functionals which are necessary to obtain an ε -approximation of I_d , i.e.,

$$n(\varepsilon, d) = \min\{n \mid e(n, d) \leq \varepsilon\}.$$

We say that the approximation problem is called weakly tractable, if

$$\lim_{\varepsilon^{-1} + d \rightarrow \infty} \frac{\ln n(\varepsilon, d)}{\varepsilon^{-1} + d} = 0,$$

i.e., $n(\varepsilon, d)$ neither depends exponentially on $1/\varepsilon$ nor on d . Otherwise, the approximation problem is called intractable.

If there exist two constants $C, t > 0$ such that for all $d \in \mathbb{N}_+$, $\varepsilon \in (0, 1)$,

$$n(\varepsilon, d) \leq C \exp(t(1 + \ln \varepsilon^{-1})(1 + \ln d)),$$

then the approximation problem is quasi-polynomially tractable.

If there exist positive numbers C, ε_0, γ such that for all $0 < \varepsilon \leq \varepsilon_0$ and infinitely many $d \in \mathbb{N}_+$,

$$n(\varepsilon, d) \geq C(1 + \gamma)^d,$$

then we say that the approximation problem suffers from the curse of dimensionality.

Recently, Siedlecki and Weimar introduced the notion of (α, β) -weak tractability in [9]. If for some fixed $\alpha, \beta > 0$ it holds

$$\lim_{\varepsilon^{-1} + d \rightarrow \infty} \frac{\ln n(\varepsilon, d)}{(\varepsilon^{-1})^\alpha + d^\beta} = 0,$$

then the approximation problem is called (α, β) -weakly tractable.

Clearly, $(1, 1)$ -weak tractability is just weak tractability, whereas the approximation problem is uniformly weakly tractable if it is (α, β) -weakly tractable for all positive α and β (see [8]). Also, if the approximation problem suffers from the curse of dimensionality, then for any $\alpha > 0$, $0 < \beta \leq 1$, it is not (α, β) -weakly tractable.

2.5. Main results.

In the paper, we discuss the asymptotic and preasymptotic behavior of the approximation numbers and tractability of the embeddings from $W_2^{\mathbf{R}}(\mathbb{T}^d)$ and $W_2^\infty(\mathbb{T}^d)$ to $L_2(\mathbb{T}^d)$. For $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$ and $t > 0$, denote by

$$B_{\mathbf{R}}^d(t) := \{\mathbf{x} \in \mathbb{R}^d : \sum_{j=1}^d |x_j|^{R_j} \leq t\}$$

the generalized ball in \mathbb{R}^d . We write $B_{\mathbf{R}}^d$ instead of $B_{\mathbf{R}}^d(1)$ for brevity. When $R_1 = \dots = R_d = r > 0$, $B_{\mathbf{R}}^d$ recedes to the unit ball B_r^d in \mathbb{R}^d with respect to the

(quasi)-norm $|\cdot|_r$. It follows from [11] that the volume of the unit ball $B_{\mathbf{R}}^d$ is

$$(2.3) \quad \text{vol}(B_{\mathbf{R}}^d) := \text{vol}\{\mathbf{x} \in \mathbb{R}^d : \sum_{j=1}^d |x_j|^{R_j} \leq 1\} = 2^d \frac{\Gamma(1+1/R_1) \cdots \Gamma(1+1/R_d)}{\Gamma(1+1/R_1 + \cdots + 1/R_d)},$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the Gamma function. Specially, we have

$$(2.4) \quad \text{vol}(B_r^d) := \text{vol}\{\mathbf{x} \in \mathbb{R}^d : \sum_{j=1}^d |x_j|^r \leq 1\} = 2^d \frac{\Gamma(1+1/r)^d}{\Gamma(1+d/r)}.$$

In [2, Theorem 1.1] or [3], the authors used entropy number argument, and combinatorial and volume arguments to obtain the preasymptotic and asymptotic behavior of the approximation numbers $a_n(I_d : H^{s,r}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d))$. For $0 < r \leq \infty$ and $s > 0$, they got

$$(2.5) \quad a_n(I_d : H^{s,r}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \asymp_{s,r} \begin{cases} 1, & 1 \leq n \leq d, \\ \left(\frac{\log(1+\frac{d}{\log n})}{\log n} \right)^{s/r}, & d \leq n \leq 2^d, \\ d^{-s/r} n^{-s/d}, & n \geq 2^d, \end{cases}$$

where $\log x = \log_2 x$, $A \asymp B$ means that there exist two constants c and C which are called the equivalent constants such that $cA \leq B \leq CA$, and $\asymp_{s,r}$ indicates that the equivalent constants depend only on s, r . It is remarkable that the equivalent constants in the above preasymptotics and asymptotics depend not on d and n . For $n \rightarrow \infty$, the equivalent constants in the lower and upper bound even converge. Indeed, in [3] (see also [1, Proposition 4.1]) the authors obtained the following result:

$$(2.6) \quad \lim_{n \rightarrow \infty} n^{s/d} a_n(I_d : H^{s,r}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) = (\text{vol}(B_r^d))^{s/d} \asymp_{s,r} d^{-s/r}.$$

In this paper, we generalize the above results to $W_2^{\mathbf{R}}(\mathbb{T}^d)$ and $W_2^\infty(\mathbb{T}^d)$. We use the volume argument to get the asymptotic behavior of $a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d))$. Note that our generalization is not trivial since we need to establish a new inequality instead of the triangle inequality of the (quasi)-norm $|\cdot|_r$. We use (2.5) to obtain the preasymptotic behavior of $a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d))$. We use combinatorial argument to obtain the preasymptotic behavior of the approximation numbers $a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d))$. Finally we give the tractability results about the approximation problems $I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ and $I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$. Our main results are formulated as follows.

Theorem 2.2. *Let $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$, $u = \max\{R_1, \dots, R_d\}$, $v = \min\{R_1, \dots, R_d\}$, $g(\mathbf{R}) = \frac{1}{1/R_1 + \dots + 1/R_d}$. Then we have*

$$(2.7) \quad a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \asymp_{u,v} \begin{cases} 1, & 1 \leq n \leq d, \\ \left(\frac{\log(1+\frac{d}{\log n})}{\log n} \right)^{1/2}, & d \leq n \leq 3^d, \\ d^{-1/2} n^{-g(\mathbf{R})}, & n \geq 3^d. \end{cases}$$

Remark 2.3. Let $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$, $u = \max\{R_1, \dots, R_d\}$, $v = \min\{R_1, \dots, R_d\}$, and $g(\mathbf{R}) = \frac{1}{1/R_1 + \dots + 1/R_d}$. Then for all $n \in \mathbb{N}_+$, the above theorem gives the exact decay rate in n and the exact order of the constants with respect to d . We emphasize that the equivalent constants in (2.7) are independent of d and n . Note that when $R_1 = \dots = R_d = s > 0$, $g(\mathbf{R}) = s/d$, the anisotropic Sobolev space $W_2^{\mathbf{R}}(\mathbb{T}^d)$ recedes to the isotropic Sobolev space $H^{s,2s}(\mathbb{T}^d)$, and (2.7) recedes to (2.5) with $r = 2s$.

Theorem 2.4. Let $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$. Then we have

$$(2.8) \quad \lim_{n \rightarrow \infty} n^{g(\mathbf{R})} a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) = (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})},$$

where $g(\mathbf{R}) = \frac{1}{1/R_1 + \dots + 1/R_d}$.

Remark 2.5. Let $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$. Then for $g(\mathbf{R}) = \frac{1}{1/R_1 + \dots + 1/R_d} > 1/2$, the space $W_2^{\mathbf{R}}(\mathbb{T}^d)$ can compactly embed into $L_\infty(\mathbb{T}^d)$ or $C(\mathbb{T}^d)$ (see [10, Theorem 3.5]). According to [1, Theorem 3.1], we know that the condition $g(\mathbf{R}) > 1/2$ is sufficient and necessary condition for the embedding from $W_2^{\mathbf{R}}(\mathbb{T}^d)$ to $L_\infty(\mathbb{T}^d)$ or $C(\mathbb{T}^d)$. Furthermore, for $g(\mathbf{R}) > 1/2$, using the proof technique of [10, Theorem 4.3] we can show

$$(2.9) \quad \lim_{n \rightarrow \infty} n^{g(\mathbf{R}) - \frac{1}{2}} a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_\infty(\mathbb{T}^d)) = (2g(\mathbf{R}) - 1)^{-\frac{1}{2}} (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})}.$$

Note that (2.9) holds if we replace $L_\infty(\mathbb{T}^d)$ with $C(\mathbb{T}^d)$.

Remark 2.6. When $R_1 = R_2 = \dots = R_d = s > 0$, (2.8) recedes to (2.6) with $r = 2s$. One can rephrase (2.8) and (2.9) as strong equivalences

$$a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \sim n^{-g(\mathbf{R})} (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})}$$

and

$$\begin{aligned} a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_\infty(\mathbb{T}^d)) &\sim (2g(\mathbf{R}) - 1)^{-\frac{1}{2}} n^{-g(\mathbf{R}) + \frac{1}{2}} (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})} \\ &\sim (2g(\mathbf{R}) - 1)^{-\frac{1}{2}} n^{\frac{1}{2}} a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)). \end{aligned}$$

The novelty of Theorem 2.4 and (2.9) is that they give strong equivalences and provide asymptotically optimal constants, for arbitrary fixed d and \mathbf{R} .

Theorem 2.7. We have

$$(2.10) \quad a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \asymp \begin{cases} 1, & 1 \leq n \leq d, \\ \left(\frac{\log(1 + \frac{d}{\log n})}{\log n} \right)^{1/2}, & d \leq n \leq 3^d, \end{cases}$$

and $a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) = 0$ for $n > 3^d$, where the equivalent constants do not depend on n and d .

Theorem 2.8. Let $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$, $\inf_{1 \leq j < \infty} R_j > 0$, and $\alpha, \beta > 0$. Then the approximation problems

$$I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d) \quad \text{and} \quad I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$$

are (α, β) -weakly tractable if and only if $\alpha > 2$ and $\beta > 0$ or $\alpha > 0$ and $\beta > 1$. Specially, the approximation problems $I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ and $I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ are intractable and do not suffer from the curse of dimensionality.

Remark 2.9. In [3], the authors considered the tractability of the isotropic Sobolev embeddings

$$(2.11) \quad I_d : H^{s,r}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$$

for the cases $r = 1$, $r = 2$, and $r = 2s$, and obtained the following results.

- (1) For every $s > 0$, none of the above mentioned approximation problems (2.11) is quasi-polynomially tractable, and suffers from the curse of dimensionality.
- (2) For $r = 1$, (2.11) is weakly tractable if $s > 1$, intractable if $0 < s \leq 1$.
- (3) For $r = 2s$, (2.11) is intractable for all $s > 0$.

(4) For $r = 2$, (2.11) is weakly tractable if $s > 2$, intractable if $0 < s \leq 1$, and remains open if $1 < s \leq 2$. Specially, the classical Sobolev embedding $I_d : H^m(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ is weakly tractable if $m \geq 3$, intractable if $m = 1$, and remains open if $m = 2$.

We remark that the above open problem was solved already in several papers [9], [2], and [12] (via a different technique).

In [9, Theorem 4.1], the authors obtained the (α, β) -weak tractability of the approximation problem (2.11) with $r = 1, r = 2s$, and $r = 2$. Combining with [2, Remark 7.6], we can see easily that the approximation problem (2.11) is (α, β) -weakly tractable for $\alpha > r/s$ and $\beta > 0$ or $\alpha > 0$ and $\beta > 1$.

Remark 2.10. It is an interesting problem about the tractability of the classical anisotropic Sobolev embedding problem $I_d : H_2^{\mathbf{m}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$, $\mathbf{m} \in \mathbb{N}_+^d$ for $k = \sup_{1 \leq j < \infty} m_j < +\infty$. Since both the embeddings $I_d : H^k(\mathbb{T}^d) \rightarrow H_2^{\mathbf{m}}(\mathbb{T}^d)$ and $I_d : H_2^{\mathbf{m}}(\mathbb{T}^d) \rightarrow H^1(\mathbb{T}^d)$ have the norm 1, it follows from [3, 2] that the approximation problem $I_d : H_2^{\mathbf{m}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ is not uniformly weakly tractable and does not suffer from the curse of dimensionality. Concerning with the weak tractability, we conjecture that the approximation problem $I_d : H_2^{\mathbf{m}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ is weakly tractable if $\liminf_{d \rightarrow \infty} d g(\mathbf{m}) > 2$, and intractable if $\liminf_{d \rightarrow \infty} d g(\mathbf{m}) \leq 2$. It is an open problem.

3. PROOFS OF THEOREMS 2.2, 2.4, 2.7, AND 2.8

For $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$, let $\{\omega_{\mathbf{R},d}^*(l)\}_{l=1}^\infty$ be the non-increasing rearrangement of

$$\left\{ \left(1 + \sum_{j=1}^d |k_j|^{2R_j} \right)^{-1/2} \right\}_{\mathbf{k} \in \mathbb{Z}^d}.$$

According to the results about approximation numbers of diagonal operators (see e.g. [3, Lemma 2.4], or [5, Corollary 4.12]), we get

$$a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) = \omega_{\mathbf{R},d}^*(n).$$

For $m \in \mathbb{N}$, denote by $C(m, \mathbf{R}, d)$ the cardinality of the set

$$\left\{ \mathbf{k} \mid \sum_{j=1}^d |k_j|^{2R_j} \leq m^{2p}, \mathbf{k} \in \mathbb{Z}^d \right\},$$

where $p = \max\{1/2, R_1, \dots, R_d\}$. Then for $C(m-1, \mathbf{R}, d) < n \leq C(m, \mathbf{R}, d)$, we have

$$(1 + m^{2p})^{-1/2} \leq a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \leq (1 + (m-1)^{2p})^{-1/2}.$$

The following lemma gives the relation of $\text{vol}(B_{\mathbf{R}}^d(t))$ ($t > 0$) and $\text{vol}(B_{\mathbf{R}}^d)$.

Lemma 3.1. *Let $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$, $t > 0$. Then*

$$(3.1) \quad \text{vol}(B_{\mathbf{R}}^d(t)) = t^{1/R_1 + \dots + 1/R_d} \text{vol}(B_{\mathbf{R}}^d).$$

Proof. We make a change of variables

$$y_1 = x_1 t^{-1/R_1}, \dots, y_d = x_d t^{-1/R_d}$$

that deforms $B_{\mathbf{R}}^d(t)$ into $B_{\mathbf{R}}^d$. The Jacobian determinant is $J(\mathbf{y}) = t^{1/R_1 + \dots + 1/R_d}$. By the change of variables formula we obtain

$$\text{vol}(B_{\mathbf{R}}^d(t)) = \int_{B_{\mathbf{R}}^d(t)} 1 \, d\mathbf{x} = \int_{B_{\mathbf{R}}^d} J(\mathbf{y}) \, d\mathbf{y} = t^{1/R_1 + \dots + 1/R_d} \text{vol}(B_{\mathbf{R}}^d),$$

which completes the proof. \square

Lemma 3.2. *Let $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$, $p = \max\{\frac{1}{2}, R_1, \dots, R_d\}$. Then for any $x, y \in \mathbb{R}^d$, we have*

$$(3.2) \quad \left(\sum_{j=1}^d |x_j + y_j|^{2R_j} \right)^{1/(2p)} \leq \left(\sum_{j=1}^d |x_j|^{2R_j} \right)^{1/(2p)} + \left(\sum_{j=1}^d |y_j|^{2R_j} \right)^{1/(2p)}$$

Proof. Using the inequality

$$(a + b)^q \leq a^q + b^q, \quad a, b \geq 0, \quad 0 < q \leq 1,$$

we get that for $1 \leq j \leq d$,

$$(3.3) \quad |x_j + y_j|^{2R_j} \leq (|x_j| + |y_j|)^{R_j/p} \leq |x_j|^{R_j/p} + |y_j|^{R_j/p}.$$

It follows from (3.3) and the Minkowskii inequality that

$$\begin{aligned} \left(\sum_{j=1}^d |x_j + y_j|^{2R_j} \right)^{1/(2p)} &\leq \left(\sum_{j=1}^d (|x_j|^{R_j/p} + |y_j|^{R_j/p})^{2p} \right)^{1/(2p)} \\ &\leq \left(\sum_{j=1}^d |x_j|^{2R_j} \right)^{1/(2p)} + \left(\sum_{j=1}^d |y_j|^{2R_j} \right)^{1/(2p)}. \end{aligned}$$

Lemma 3.2 is proved. \square

Lemma 3.3. ([3]) *For $0 \leq x < \infty$, it holds*

$$(3.4) \quad \left(\frac{x}{e} \right)^x \leq \Gamma(1+x) \leq (1+x)^x.$$

Note that $dg(\mathbf{R}) = \frac{d}{1/R_1 + \dots + 1/R_d}$ is just the harmonic average of the d positive numbers R_1, \dots, R_d and is between R_1, \dots, R_d .

Lemma 3.4. *Let $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$, $u = \max\{R_1, \dots, R_d\}$, $v = \min\{R_1, \dots, R_d\}$. Then for all $d \in \mathbb{N}_+$, we have*

$$(3.5) \quad 2^v \sqrt{\frac{1}{e(d+2u)}} \leq (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})} \leq 2^u \left(\frac{2v+1}{2v} \right)^{\frac{u}{2v}} \sqrt{\frac{2eu}{d}},$$

where $g(\mathbf{R}) = \frac{1}{1/R_1 + \dots + 1/R_d}$, $v \leq dg(\mathbf{R}) \leq u$.

Proof. By (2.3), we get

$$\begin{aligned} \left(\text{vol}(B_{2\mathbf{R}}^d) \right)^{g(\mathbf{R})} &= 2^{dg(\mathbf{R})} \frac{\left(\Gamma(1 + \frac{1}{2R_1}) \cdots \Gamma(1 + \frac{1}{2R_d}) \right)^{g(\mathbf{R})}}{\left(\Gamma(1 + \frac{1}{2R_1} + \dots + \frac{1}{2R_d}) \right)^{g(\mathbf{R})}} \\ &= 2^{dg(\mathbf{R})} \left(\Gamma(1 + \frac{1}{2R_1}) \cdots \Gamma(1 + \frac{1}{2R_d}) \right)^{g(\mathbf{R})} \left(\Gamma(1 + \frac{1}{2g(\mathbf{R})}) \right)^{-g(\mathbf{R})}. \end{aligned}$$

From Lemma 3.3, we have

$$(3.6) \quad \left(\Gamma\left(1 + \frac{1}{2g(\mathbf{R})}\right) \right)^{-g(\mathbf{R})} \leq \left(\frac{1}{2eg(\mathbf{R})} \right)^{-\frac{1}{2}} = \left(\frac{2edg(\mathbf{R})}{d} \right)^{\frac{1}{2}},$$

and

$$(3.7) \quad \left(\Gamma\left(1 + \frac{1}{2g(\mathbf{R})}\right) \right)^{-g(\mathbf{R})} \geq \left(1 + \frac{1}{2g(\mathbf{R})} \right)^{-\frac{1}{2}} = \left(\frac{2dg(\mathbf{R})}{d + 2dg(\mathbf{R})} \right)^{\frac{1}{2}}.$$

It follows from Lemma 3.3 and the monotonicity of the function $g(x) = (1+x)^x$, $x \geq 0$ that

$$(3.8) \quad \begin{aligned} \left(\Gamma\left(1 + \frac{1}{2R_1}\right) \cdots \Gamma\left(1 + \frac{1}{2R_d}\right) \right)^{g(\mathbf{R})} &\leq \left(\left(1 + \frac{1}{2R_1}\right)^{\frac{1}{2R_1}} \cdots \left(1 + \frac{1}{2R_d}\right)^{\frac{1}{2R_d}} \right)^{g(\mathbf{R})} \\ &\leq \left(1 + \frac{1}{2v} \right)^{\frac{dg(\mathbf{R})}{2v}}. \end{aligned}$$

Next we use the convexity of the function $\ln \Gamma(x)$, $x > 0$ to get that for $x_1, \dots, x_d \geq 0$,

$$\Gamma(1 + x_1) \cdots \Gamma(1 + x_d) \geq \left(\Gamma\left(1 + \frac{x_1 + \cdots + x_d}{d}\right) \right)^d.$$

It follows that

$$(3.9) \quad \left(\Gamma\left(1 + \frac{1}{2R_1}\right) \cdots \Gamma\left(1 + \frac{1}{2R_d}\right) \right)^{g(\mathbf{R})} \geq \left(\Gamma\left(1 + \frac{1}{2dg(\mathbf{R})}\right) \right)^{dg(\mathbf{R})} \geq \left(\frac{1}{2edg(\mathbf{R})} \right)^{\frac{1}{2}}.$$

Then (3.6) and (3.8) lead to

$$(\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})} \leq 2^{dg(\mathbf{R})} \left(1 + \frac{1}{2v} \right)^{\frac{dg(\mathbf{R})}{2v}} \left(\frac{2edg(\mathbf{R})}{d} \right)^{\frac{1}{2}},$$

and (3.7) and (3.9) yield

$$(\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})} \geq 2^{dg(\mathbf{R})} \left(\frac{1}{e(d + 2dg(\mathbf{R}))} \right)^{\frac{1}{2}}.$$

Hence, (3.5) follows from the two above inequalities and $v \leq dg(\mathbf{R}) \leq u$ immediately. Lemma 3.4 is proved. \square

Lemma 3.5. *Let $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$, $u = \max\{R_1, \dots, R_d\}$, $v = \min\{R_1, \dots, R_d\}$, $p = \max\{\frac{1}{2}, u\}$, $g(\mathbf{R}) = \frac{1}{1/R_1 + \dots + 1/R_d}$. Then for $n > E^d$, we have*

$$(3.10) \quad n^{g(\mathbf{R})} a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \leq 2^{(p+u)} \left(1 + \frac{2v+1}{2v} \right)^{\frac{u}{2v}} \sqrt{\frac{2eu}{d}},$$

and

$$(3.11) \quad n^{g(\mathbf{R})} a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \geq 2^{(v-p)} \sqrt{\frac{1}{e(d + 2u)}},$$

where $E := 4^{p/v} 2^{u/v} (1 + 1/(2v))^{u/(2v^2)} (2ep)^{1/(2v)}$ is a positive constant depending only on u and v .

Proof. For any $m \in \mathbb{N}_+$, let $Q_{\mathbf{k}}$ be a cube with center \mathbf{k} , sides parallel to the axes and side-length 1. It follows from (3.2) that

$$\begin{aligned} B_{2\mathbf{R}}^d \left(\left(m - \left(\sum_{j=1}^d \left(\frac{1}{2} \right)^{2R_j} \right)^{1/(2p)} \right)_+^{2p} \right) &\subset \bigcup_{\substack{\mathbf{k} \in \mathbb{Z}^d \\ \sum_{j=1}^d |k_j|^{2R_j} \leq m^{2p}}} Q_{\mathbf{k}} \\ &\subset B_{2\mathbf{R}}^d \left(\left(m + \left(\sum_{j=1}^d \left(\frac{1}{2} \right)^{2R_j} \right)^{1/(2p)} \right)^{2p} \right), \end{aligned}$$

where a_+ is equal to a if $a \geq 0$ and 0 if $a < 0$. Note that the volume of the set

$$\bigcup_{\substack{\mathbf{k} \in \mathbb{Z}^d \\ \sum_{j=1}^d |k_j|^{2R_j} \leq m^{2p}}} Q_{\mathbf{k}}$$

is just $C(m, \mathbf{R}, d)$. By (3.1) we get

$$\begin{aligned} (3.12) \quad & \left(m - \left(\sum_{j=1}^d \left(\frac{1}{2} \right)^{2R_j} \right)^{1/(2p)} \right)_+^{p/g(\mathbf{R})} \text{vol}(B_{2\mathbf{R}}^d) \leq C(m, \mathbf{R}, d) \\ & \leq \left(m + \left(\sum_{j=1}^d \left(\frac{1}{2} \right)^{2R_j} \right)^{1/(2p)} \right)^{p/g(\mathbf{R})} \text{vol}(B_{2\mathbf{R}}^d). \end{aligned}$$

We set

$$a_n(I_d) := a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \quad \text{and} \quad b_{\mathbf{R}} := \left(\sum_{j=1}^d \left(\frac{1}{2} \right)^{2R_j} \right)^{1/(2p)}.$$

We also set

$$A(m, \mathbf{R}, d) := (m + b_{\mathbf{R}})^{p/g(\mathbf{R})} \text{vol}(B_{2\mathbf{R}}^d)$$

and

$$B(m, \mathbf{R}, d) := (m - b_{\mathbf{R}})_+^{p/g(\mathbf{R})} \text{vol}(B_{2\mathbf{R}}^d).$$

First we estimate the upper bound for $n^{g(\mathbf{R})} a_n(I_d)$. We fixed $m_0 \in \mathbb{N}$ such that $1 + b_{\mathbf{R}} \leq m_0 \leq 2 + b_{\mathbf{R}}$. For all $m \geq m_0$, suppose that

$$A(m, \mathbf{R}, d) < n \leq A(m+1, \mathbf{R}, d).$$

By (3.12) we have $n > C(m, \mathbf{R}, d)$, which implies $a_n(I_d) \leq (1 + m^{2p})^{-\frac{1}{2}}$. It follows that

$$\begin{aligned} n^{g(\mathbf{R})} a_n(I_d) &\leq \frac{(A(m+1, \mathbf{R}, d))^{g(\mathbf{R})}}{(1 + m^{2p})^{\frac{1}{2}}} = \frac{(m+1 + b_{\mathbf{R}})^p (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})}}{(1 + m^{2p})^{\frac{1}{2}}} \\ &\leq \frac{(m+1 + b_{\mathbf{R}})^p (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})}}{m^p} \leq 2^p (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})}. \end{aligned}$$

By (3.5) we get further

$$n^{g(\mathbf{R})} a_n(I_d) \leq 2^{(p+u)} \left(\frac{2v+1}{2v} \right)^{\frac{u}{2v}} \sqrt{\frac{2eu}{d}},$$

which proves (3.10) for large enough $n > A(m_0, \mathbf{R}, d)$. Note that $b_{\mathbf{R}} \leq d^{1/(2p)}$. Using (3.5) we get that

$$\begin{aligned} A(m_0, \mathbf{R}, d) &\leq (2 + 2b_{\mathbf{R}})^{p/g(\mathbf{R})} \text{vol}(B_{2\mathbf{R}}^d) \leq (4d^{1/(2p)})^{p/g(\mathbf{R})} \text{vol}(B_{2\mathbf{R}}^d) \\ &\leq (4d^{1/(2p)})^{p/g(\mathbf{R})} \left(2^u (1 + 1/(2v))^{u/(2v)} (2eu)^{1/2} d^{-1/2} \right)^{1/g(\mathbf{R})} \\ &\leq \left(4^p 2^u (1 + 1/(2v))^{u/(2v)} (2ep)^{1/2} \right)^{d/(dg(\mathbf{R}))} \\ &\leq \left(4^{p/v} 2^{u/v} (1 + 1/(2v))^{u/(2v^2)} (2ep)^{1/(2v)} \right)^d = E^d. \end{aligned}$$

Hence, (3.10) holds for $n > E^d$.

Next we show the lower bound. Choose $m_1 \in \mathbb{N}$ such that $2 + 2b_{\mathbf{R}} \leq m_1 \leq 3 + 2b_{\mathbf{R}}$. For $m \geq m_1$ and

$$B(m, \mathbf{R}, d) < n \leq B(m + 1, \mathbf{R}, d),$$

by (3.12) we have $n \leq C(m + 1, \mathbf{R}, d)$, which means $a_n(I_d) \geq (1 + (m + 1)^{2p})^{-\frac{1}{2}}$. It follows that

$$\begin{aligned} n^{g(\mathbf{R})} a_n(I_d) &\geq \frac{(B(m, \mathbf{R}, d))^{g(\mathbf{R})}}{(1 + (m + 1)^{2p})^{\frac{1}{2}}} = \frac{(m - b_{\mathbf{R}})^p (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})}}{(1 + (m + 1)^{2p})^{\frac{1}{2}}} \\ &\geq \frac{(m - b_{\mathbf{R}})^p (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})}}{(m + 2)^p} \geq \left(\frac{2 + b_{\mathbf{R}}}{4 + 2b_{\mathbf{R}}} \right)^p (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})} \\ &= 2^{-p} \text{vol}(B_{2\mathbf{R}}^d)^{g(\mathbf{R})} \geq 2^{(v-p)} (e(d + 2u))^{-1/2}, \end{aligned}$$

where in the last second inequality we used the monotonicity of the function $\frac{x-b_{\mathbf{R}}}{x+2}$, $x \geq 0$, and in the last inequality we used (3.5). Hence (3.11) holds for $n > B(m_1, \mathbf{R}, d)$. Similarly, we have

$$\begin{aligned} B(m_1, \mathbf{R}, d) &\leq (3 + b_{\mathbf{R}})^{p/g(\mathbf{R})} \text{vol}(B_{2\mathbf{R}}^d) \leq (4d^{1/(2p)})^{p/g(\mathbf{R})} \text{vol}(B_{2\mathbf{R}}^d) \\ &\leq \left(4^{p/v} 2^{u/v} (1 + 1/(2v))^{u/(2v^2)} (2ep)^{1/(2v)} \right)^d = E^d. \end{aligned}$$

This means that (3.11) holds for $n > E^d$. The proof of Lemma 3.5 is completed. \square

Remark 3.6. Let $\mathbf{R} = (R_1, \dots, R_d) \in \mathbb{R}_+^d$, $u = \max\{R_1, \dots, R_d\}$, $v = \min\{R_1, \dots, R_d\}$, and $g(\mathbf{R}) = \frac{1}{1/R_1 + \dots + 1/R_d}$. Then for sufficiently large n ($n > E^d$), the above lemma provides the two-sides inequalities

$$(3.13) \quad c_{u,v} d^{-1/2} n^{-g(\mathbf{R})} \leq a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \leq C_{u,v} d^{-1/2} n^{-g(\mathbf{R})},$$

where the constants $c_{u,v}, C_{u,v}$ depend only on u and v . Note that we captured the exact decay rate in n and the exact order of the constants with respect to d .

Proof of Theorem 2.2.

We note that both the embeddings

$$W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow H^{v, 2v}(\mathbb{T}^d) \quad \text{and} \quad H^{u, 2u}(\mathbb{T}^d) \rightarrow W_2^{\mathbf{R}}(\mathbb{T}^d)$$

have norm 1. It follows that

$$a_n(I_d : H^{u, 2u}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \leq a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \leq a_n(I_d : H^{v, 2v}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)).$$

Noting

$$\frac{\log n}{\log(1 + \frac{d}{\log n})} \asymp d \text{ for } 2^d \leq n \leq 3^d,$$

and

$$n^{-u/d} \asymp_{u,v} n^{-v/d} \asymp_{u,v} n^{-g(\mathbf{R})} \asymp_{u,v} 1 \text{ for } 3^d \leq n \leq E^d,$$

we obtain from (2.5) that

$$\begin{aligned} a_n(I_d : H^{u,2u}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) &\asymp_{u,v} a_n(I_d : H^{v,2v}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \\ &\asymp_{u,v} \begin{cases} 1, & 1 \leq n \leq d, \\ \left(\frac{\log(1 + \frac{d}{\log n})}{\log n}\right)^{1/2}, & d \leq n \leq 3^d, \\ d^{-1/2}, & 3^d \leq n \leq E^d, \end{cases} \end{aligned}$$

where E is a positive constant given in Lemma 3.5 which depends only on u and v . We have

$$a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \asymp_{u,v} \begin{cases} 1, & 1 \leq n \leq d, \\ \left(\frac{\log(1 + \frac{d}{\log n})}{\log n}\right)^{1/2}, & d \leq n \leq 3^d, \\ d^{-1/2} n^{-g(\mathbf{R})}, & 3^d \leq n \leq E^d, \end{cases}$$

which combining with (3.13), yields (2.7). Theorem 2.2 is proved. \square

Proof of Theorem 2.4.

As in the proof of Lemma 3.5, we set

$$a_n(I_d) := a_n(I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \text{ and } b_{\mathbf{R}} := \left(\sum_{j=1}^d \left(\frac{1}{2}\right)^{2R_j}\right)^{1/(2p)}.$$

For $C(m-1, \mathbf{R}, d) < n \leq C(m, \mathbf{R}, d)$, we have

$$(1 + m^{2p})^{-1/2} \leq a_n(I_d) \leq (1 + (m-1)^{2p})^{-1/2}.$$

It follows that

$$(3.14) \quad n \cdot (a_n(I_d))^{1/g(\mathbf{R})} \leq C(m, \mathbf{R}, d)(1 + (m-1)^{2p})^{-1/(2g(\mathbf{R}))},$$

and

$$(3.15) \quad n \cdot (a_n(I_d))^{1/g(\mathbf{R})} > C(m-1, \mathbf{R}, d)(1 + m^{2p})^{-1/(2g(\mathbf{R}))}.$$

On one side, from (3.12) and (3.14) we get

$$n \cdot (a_n(I_d))^{1/g(\mathbf{R})} \leq (1 + (m-1)^{2p})^{-1/2g(\mathbf{R})} (m + b_{\mathbf{R}})^{p/g(\mathbf{R})} \text{vol}(B_{2\mathbf{R}}^d).$$

Obviously, it holds that

$$\lim_{m \rightarrow \infty} (1 + (m-1)^{2p})^{-1/2g(\mathbf{R})} (m + b_{\mathbf{R}})^{p/g(\mathbf{R})} = 1,$$

which implies that

$$\lim_{n \rightarrow \infty} n^{g(\mathbf{R})} a_n(I_d) \leq (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})}.$$

On the other side, (3.12) and (3.15) lead to

$$n \cdot (a_n(I_d))^{1/g(\mathbf{R})} > (1 + m^{2p})^{-1/2g(\mathbf{R})} (m - b_{\mathbf{R}})_+^{p/g(\mathbf{R})} \text{vol}(B_{2\mathbf{R}}^d).$$

Since

$$\lim_{m \rightarrow \infty} (1 + m^{2p})^{-1/2g(\mathbf{R})} (m - b_{\mathbf{R}})_+^{p/g(\mathbf{R})} = 1,$$

we obtain

$$\lim_{n \rightarrow \infty} n^{g(\mathbf{R})} a_n(I_d) \geq (\text{vol}(B_{2\mathbf{R}}^d))^{g(\mathbf{R})}.$$

Theorem 2.4 is proved. \square

Proof of Theorem 2.7.

First we note that $\dim(W_2^\infty(\mathbb{T}^d)) = 3^d$, which implies

$$a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) = 0$$

for $n > 3^d$. Let $\{\omega_d^*(l)\}_{1 \leq l \leq 3^d}$ be the non-increasing rearrangement of

$$\left\{ \left(1 + \sum_{j=1}^d |k_j| \right)^{-1/2} \right\}_{\mathbf{k} \in \{-1, 0, 1\}^d}.$$

Then we have for $1 \leq n \leq 3^d$,

$$a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) = \omega_d^*(n).$$

For $m = 0, 1, \dots, d$, denote by $C(m, d)$ and $D(m, d)$ the cardinalities of the sets

$$\left\{ \mathbf{k} \in \{-1, 0, 1\}^d \mid \sum_{j=1}^d |k_j| \leq m \right\} \quad \text{and} \quad \left\{ \mathbf{k} \in \{-1, 0, 1\}^d \mid \sum_{j=1}^d |k_j| = m \right\}.$$

Then

$$a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) = (1 + m)^{-1/2}$$

for $C(m-1, d) < n \leq C(m, d)$, $m = 1, \dots, d$. It is easy to see that for $m = 0, 1, \dots, d$,

$$D(m, d) = 2^m \binom{d}{m} \quad \text{and} \quad C(m, d) = \sum_{j=0}^m D(j, d) = \sum_{j=0}^m 2^j \binom{d}{j}.$$

For $1 \leq n \leq C(2, d) = 2d^2 + 1$, we have

$$3^{-1/2} \leq a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \leq 1.$$

This means that

$$a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \asymp 1 \asymp \begin{cases} 1, & 1 \leq n \leq d, \\ \left(\frac{\log(1 + \frac{d}{\log n})}{\log n} \right)^{1/2}, & d \leq n \leq C(2, d). \end{cases}$$

For $C(m, d) < n \leq 3^d$, $d/2 \leq m \leq d$, we have

$$d/2 \leq m \leq \log C(m, d) \leq \log n \leq d \log 3,$$

and

$$\begin{aligned} (1 + d)^{-1/2} &= a_{3^d}(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \\ &\leq a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \\ &\leq (2 + m)^{-1/2} \leq (2 + d/2)^{-1/2}, \end{aligned}$$

which implies that for $C(m, d) < n \leq 3^d$, $d/2 \leq m \leq d$,

$$a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \asymp d^{-1/2} \asymp \left(\frac{\log(1 + \frac{d}{\log n})}{\log n} \right)^{1/2}.$$

For $C(m-1, d) < n \leq C(m, d)$, $3 \leq m < d/2$, we have

$$(3.16) \quad a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) = (1+m)^{-1/2}.$$

We note that

$$n \leq \sum_{j=0}^m 2^j \binom{d}{j} \leq (m+1)2^m \binom{d}{m} \leq 2^{2m} e^m \left(\frac{d}{m}\right)^m,$$

where in the above second inequality we used the inequality $\binom{d}{j} \leq \binom{d}{m}$ for $0 \leq j \leq m < d/2$, in the above last inequality we used the inequality (see [3, (3.6)])

$$\binom{m+d}{d} \leq e^{d-1} (1+m/d)^d \leq e^d \left(\frac{m+d}{d}\right)^d.$$

It follows that $\log n \leq m \log(4ed/m)$, which implies

$$m \geq \frac{\log n}{\log(4ed/m)}.$$

Using the inequalities $\log n \leq m \log(4ed/m)$ and $x \geq 2 \log x$ for $x \geq 2$, we obtain

$$\log \left(\frac{4ed}{\log n} \right) \geq \log \left(\frac{4ed}{m \log(4ed/m)} \right) = \log \left(\frac{4ed}{m} \right) - \log \left(\log \left(\frac{4ed}{m} \right) \right) \geq \frac{1}{2} \log \left(\frac{4ed}{m} \right).$$

This yields

$$(3.17) \quad m \geq \frac{\log n}{2 \log(4ed/(\log n))}.$$

On other hand, using the inequality (see [3, (3.5)])

$$\binom{m+d}{m} \geq \max \left\{ \left(\frac{d+m}{m} \right)^m, \left(\frac{d+m}{d} \right)^d \right\},$$

we have

$$n > C(m-1, d) \geq 2^{m-1} \binom{d}{m-1} \geq 2^{m-1} \left(\frac{d}{m-1} \right)^{m-1}.$$

This yields

$$m-1 \leq \frac{\log n}{\log \left(\frac{2d}{m-1} \right)} \leq \log n.$$

It follows that

$$m \leq \frac{\log n}{\log \left(\frac{2d}{\log n} \right)} + 1,$$

which combining with (3.17) and (3.16), leads to

$$a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \asymp m^{-1/2} \asymp \left(\frac{\log n}{\log \left(\frac{2d}{\log n} \right)} \right)^{-1/2} \asymp \left(\frac{\log(1 + \frac{d}{\log n})}{\log n} \right)^{1/2}.$$

Theorem 2.7 is proved. \square

Remark 3.7. The upper estimate of $a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d))$ can also be obtained directly by (2.5) and the inequality

$$a_n(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \leq a_n(I_d : W_2^1(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)).$$

Proof of Theorem 2.8.

We set $q = \inf_{1 \leq j < \infty} R_j > 0$. Both the embeddings

$$W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow H^{q,2q}(\mathbb{T}^d) \quad \text{and} \quad W_2^\infty(\mathbb{T}^d) \rightarrow W_2^{\mathbf{R}}(\mathbb{T}^d)$$

have norm 1.

We note that the approximation problem

$$I_d : H^{s,2s}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$$

is (α, β) -weakly tractable if $\alpha > 0$ and $\beta > 1$ (see [9, Theorem 4.1]) or $\alpha > 2$ and $\beta > 0$ (see [2, Remark 7.6]) for any $s > 0$. This means that if $\alpha > 0$ and $\beta > 1$ or $\alpha > 2$ and $\beta > 0$, the approximation problems

$$I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d) \quad \text{and} \quad I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$$

are (α, β) -weakly tractable.

On the other hand, it suffices to prove the (α, β) -weak intractability of the approximation problems $I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ for $0 < \alpha \leq 2$ and $0 < \beta \leq 1$.

We note that

$$e(n, d) = a_{n+1}(I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)).$$

This means that $e(3^d - 1, d) = (1 + d)^{-1/2}$ and $e(n, d) = 0$ for $n \geq 3^d$. Choose

$$\varepsilon = \varepsilon_d = (2 + d)^{-1/2}.$$

Then

$$n(\varepsilon_d, d) = \inf\{n \in \mathbb{N} \mid e(n, d) \leq \varepsilon_d\} = 3^d.$$

If $0 < \alpha \leq 2$ and $0 < \beta \leq 1$, then we have

$$\lim_{1/\varepsilon_d + d \rightarrow \infty} \frac{\ln(n(\varepsilon_d, d))}{(\varepsilon_d)^{-\alpha} + d^\beta} \geq \lim_{d \rightarrow \infty} \frac{d \ln 3}{d + 2 + d} = \frac{\ln 3}{2} \neq 0,$$

which implies that the approximation problems $I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ is not (α, β) -weakly tractable if $0 < \alpha \leq 2$ and $0 < \beta \leq 1$.

Hence, the approximation problems

$$I_d : W_2^{\mathbf{R}}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d) \quad \text{and} \quad I_d : W_2^\infty(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$$

are (α, β) -weakly tractable if and only if $\alpha > 0$ and $\beta > 1$ or $\alpha > 2$ and $\beta > 0$. The proof of Theorem 2.8 is finished. \square

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